SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014

YEAR 11

ASSESSMENT TASK #1

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Question is to be returned in a separate bundle.
- All necessary working should be shown in every question.
- Answers should be in simplest exact from unless specified otherwise.

Total Marks – 60

- Attempt questions 1 – 4.
- Each question is worth 15 marks.

Examiner: *E. Choy & A. Fuller*

Question One (15 marks)

(a) By considering the expansion of $\cos(\alpha - \beta)$, find the exact value of $\cos 15^\circ$. [2]

(b) Find the coordinates of the point P which divides the interval from $A(-1,5)$ to $B(6,-4)$ externally in the ratio $2 : 3$. [2]

(c) The acute angle between the lines $x + y = 3$ and $3x - y = 1$ is θ. What is the value of $\tan \theta$? [2]

(d) If α, β, γ are the roots of $2x^3 - 4x^2 - 3x - 1 = 0$, find the value of the following: [7]

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$

(iii) $\alpha\beta\gamma$

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(v) $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(vi) $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$

(e) Using the *t*-results, or otherwise, prove that [2]

$\cot\frac{\theta}{2} - \cot\theta = \operatorname{cosec}\theta.$

Question Two (15 marks) **(Use a SEPARATE writing booklet)**

(a) (i) Show that $(x+2)$ is a factor of $P(x) = x^3 - 2x^2 - 20x - 24$. [3]

(ii) Hence, fully factorise $P(x) = x^3 - 2x^2 - 20x - 24$.

(b) Find the general solution of $2\cos 2x = 1$. [2]

(c) Find all values of x such that $x + \frac{36}{x} \geq 13$. [3]

(d) By substituting $y = \sin x + \cos x$ into $\sin 2x + 1 = 5\sin x + 5\cos x$, [3]

(i) show that $y^2 - 5y = 0$.

(ii) Hence, or otherwise, find all solutions between 0 and 2π of

$\sin 2x + 1 = 5\sin x + 5\cos x$.

(e) (i) By repeated substitution of $x = 1$, and differentiation, [4]

or otherwise, find constants a, b, c, d and e such that

$$x^4 - 4x^3 + x^2 + 6x + 2 \equiv a(x-1)^4 + b(x-1)^3 + c(x-1)^2 + d(x-1) + e$$

(ii) Hence, or otherwise, solve $x^4 - 4x^3 + x^2 + 6x + 2 = 0$

Question Three (15 marks) **(Use a SEPARATE writing booklet)**

(a) The point $A(a, c)$ lies on the line $4x + 3y - 30 = 0$ and the point $B(b, d)$ lies on the line $x + 3y + 15 = 0$. [6]

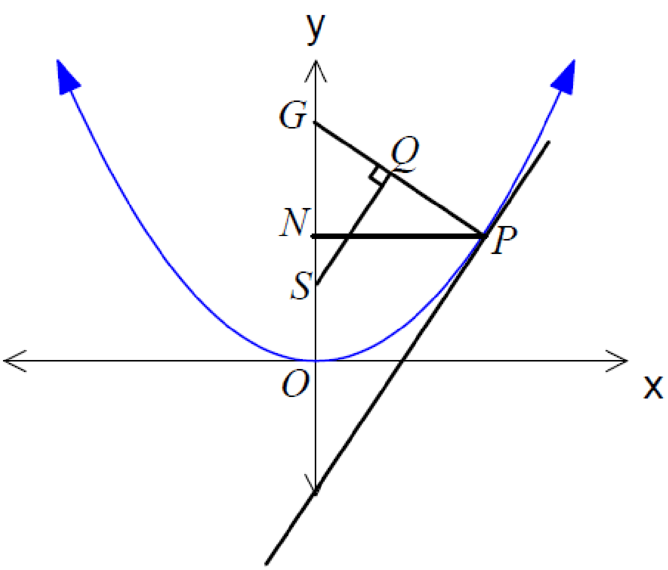
If $P(3,2)$ divides the interval AB in the ratio $2 : 1$.

(i) Show that $a = 9 - 2b$ and $c = 6 - 2d$.

(ii) Explain why $b + 3d + 15 = 0$.

(iii) Find the coordinates of A and B.

(b) Consider the variable point $P(2at, at^2)$ on the parabola $x^2 = 4ay$. [9]



(i) Show that the equation of the normal at P is $x + ty = at^3 + 2at$

(ii) The line from S, the focus of the parabola, meets the normal at P at right angles at Q, as shown.

Show that Q has coordinates $(at, a(t^2 + 1))$.

(iii) Show that the locus of Q is a parabola and state its focal length.

(iv) The line from P, parallel to the x-axis, meets the y-axis at N.

By finding the perpendicular distance of the focus to the normal, or otherwise, show that $SQ^2 = ON \times SP$.

Question Four (15 marks) **(Use a SEPARATE writing booklet)**

(a) Without the use of calculus, find the range of the following: [5]

(i) $y = \sin^2 x - 2\sin x + 2$

(ii) $y = \cos x - 2\sin x + 2$

(b) Given a parabola $P : y^2 = 4x$. [10]

Let the line $L : y = mx + c$ be a tangent to P.

(i) Show that $c = \frac{1}{m}$.

(ii) Suppose that L passes through the point (h, k). Using the above result, show that $hm^2 - km + 1 = 0$.

(iii) If the slopes of the two tangents from the point (h, k) to the parabola P are m_1 and m_2, show that $(m_1 - m_2)^2 = \frac{k^2 - 4h}{h^2}$.

(iv) Q is a point on the circle $x^2 + y^2 = 20$. The slopes of the two tangents from Q to P are s_1 and s_2 such that $\left(\frac{1}{s_2} - \frac{1}{s_1}\right)^2 = 8$.

Find the coordinates of Q.

(v) R is a point such that the angle between the two tangents from R to P is 45°. Find the equation of the locus of R.

End of Examination

2014 Mathematics Extension Task 1: Solutions

Question 1 (15 marks)

(a) By considering the expansion of $\cos(\alpha - \beta)$, find the exact value of $\cos 15^\circ$. [2]

Solution: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\ \left[\text{or}\ \cos(60^\circ - 45^\circ)\right],\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\ \left[\text{or}\ \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ\right],\\ &= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}\ \left[\text{or}\ \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}\right],\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\ \text{or}\ \frac{\sqrt{2}(\sqrt{3}+1)}{4}.\end{aligned}$$

(b) Find the coordinates of the point P which divides the interval from $A(-1, 5)$ to $B(6, -4)$ externally in the ratio $2 : 3$. [2]

Solution: $$x_P = \frac{3(-1) - 2(6)}{3-2} = -15 \qquad y_P = \frac{3(5) - 2(-4)}{3-2} = 23.$$

So P is $(-15, 23)$.

(c) The acute angle between the lines $x + y = 3$ and $3x - y = 1$ is θ. What is the value of $\tan\theta$? [2]

Solution:
$$\begin{aligned} y &= -x + 3, \text{ so } m_1 = -1,\\ y &= 3x - 1, \text{ so } m_2 = 3,\\ \therefore \tan\theta &= \frac{|-1-3|}{1+(-1)3},\\ &= 2.\end{aligned}$$

(d) If α, β, γ are the roots of $2x^3 - 4x^2 - 3x - 1 = 0$, find the value of the following: [7]

(i) $\alpha + \beta + \gamma$

Solution: $$\alpha + \beta + \gamma = -\left(\frac{-4}{2}\right), = 2.$$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$

Solution: $\alpha\beta + \alpha\gamma + \beta\gamma = -\dfrac{3}{2}$.

(iii) $\alpha\beta\gamma$

Solution: $$\begin{aligned}\alpha\beta\gamma &= -\left(\frac{-1}{2}\right),\\ &= \frac{1}{2}.\end{aligned}$$

(iv) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

Solution: $$\begin{aligned}\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} &= \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma},\\ &= -\frac{3}{2}\div\frac{1}{2},\\ &= -3.\end{aligned}$$

(v) $(\alpha-1)(\beta-1)(\gamma-1)$

Solution:
$$\begin{aligned}(\alpha-1)(\beta-1)(\gamma-1) &= \alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-1,\\ &= \tfrac{1}{2}+\tfrac{3}{2}+2-1,\\ &= 3.\end{aligned}$$

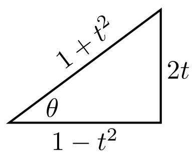
(vi) $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$

Solution:
$$\begin{aligned}(\alpha\beta+\alpha\gamma+\beta\gamma)^2 &= \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2+2\big(\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta\gamma^2\big),\\ \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 &= (\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma),\\ &= \frac{9}{4}-2.\frac{1}{2}.2,\\ &= \frac{1}{4}.\end{aligned}$$

(e) Using the t-results, or otherwise, prove that $\cot\frac{\theta}{2}-\cot\theta=\operatorname{cosec}\theta$. 2

Solution: Put $\tan\frac{\theta}{2} = t$.

$$\begin{aligned}\text{L.H.S.} &= \frac{1}{t}-\frac{1-t^2}{2t},\\ &= \frac{2-1+t^2}{2t},\\ &= \frac{1+t^2}{2t},\\ &= \frac{1}{\sin\theta},\\ &= \text{R.H.S.}\end{aligned}$$



QUESTION 2.

(a) (i). $P(-2) = -8 - 8 + 40 - 24$
$= 0$.

$\therefore x+2$ is a factor. [1]

(ii)
$$\begin{array}{r} x^2 - 4x - 12 \\ x+2 \enclose{longdiv}{x^3 - 2x^2 - 20x - 24} \\ \underline{x^3 + 2x^2} \qquad\qquad \\ -4x^2 - 20x \qquad \\ \underline{-4x^2 + 8x} \qquad \\ -12x - 24 \\ \underline{-12x - 24} \\ 0 \end{array}$$

$\therefore P(x) = (x+2)(x^2 - 4x - 12)$
$= (x+2)(x-6)(x+2)$
$= (x+2)^2(x-6)$ [2]

OR.

$$\begin{array}{r|rrrr} & 1 & -2 & -20 & -24 \\ & & -2 & 8 & 24 \\ -2 & & & & \\ \hline & 1 & -4 & -12 & 0 \end{array}$$

$\therefore (x+2)(x^2 - 4x - 12) = (x+2)^2(x-6)$

OR. $x^3 - 2x^2 - 20x - 24 = (x+2)(x^2 + bx + c)$

Clearly $2c = -24$
$c = -12$.

& co-eff of x is $2b + c = -20$
$2b - 12 = -20$
$2b = -8$
$b = -4$

$\therefore (x+2)(x^2 - 4x - 12) = (x+2)^2(x-6)$.

(b) $2\cos 2x = 1.$

$\cos 2x = \frac{1}{2}$

$2x = 2n\pi \pm \frac{\pi}{3} \quad ; \ n \in \mathbb{Z}.$

$x = n\pi \pm \frac{\pi}{6} \quad ; \ n \in \mathbb{Z}.$ [2]

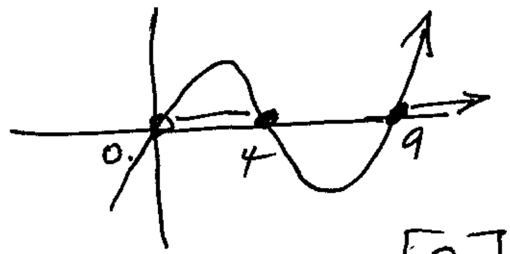
(c) $x + \frac{36}{x} \geq 13.$

$\therefore x^3 + 36x \geq 13x^2$ ($\times$ both sides by x^2)

$x^3 - 13x^2 + 36x \geq 0.$

$x(x^2 - 13x + 36) \geq 0$

$x(x-4)(x-9) \geq 0.$



$\therefore$ $0 < x \leq 4, \quad x \geq 9$

[3]

(d) (i) $\sin 2x + 1 = 2\sin x\cos x + \sin^2 x + \cos^2 x$.

$= (\sin x + \cos x)^2$.

$\therefore\ y^2 = 5y$.

OR $y^2 - 5y = 0$.

[2]

(ii) $\therefore$ if $y^2 - 5y = 0$

$y = 0$ or 5

ie. $\sin x + \cos x = 0$ OR $\sin x + \cos x = 5$

$\sin x = -\cos x$ NO. SOLNS!!

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$ [1]

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

(e) Now $x^4 - 4x^3 + x^2 + 6x + 2 \equiv a(x-1)^4 + b(x-1)^3 + c(x-1)^2 + d(x-1) + e$ (A)

Clearly MONIC $\therefore$ $a = 1$

Also if $x = 1$

$1 - 4 + 1 + 6 + 2 = e$

$e = 6$

Differentiate both sides of (A) where $a = 1$

$4x^3 - 12x^2 + 2x + 6 \equiv 4(x-1)^3 + 3b(x-1)^2 + 2c(x-1) + d$ (B)

let $x = 1$

(e) (CONTD)

$4-12+2+6 = d.$

$d = 0.$

Differentiate (B)

$12x^2-24x+2 = 12(x-1)^2 + 6b(x-1) + 2c.$ (C)

let $x=1$

$12-24+2 = 2c$

$\boxed{c=-5}$

Differentiate (C) [2]

$24x-24 = 24(x-1)+6b$

let $x=1$

$0 = 6b$

$\boxed{b=0.}$

$\therefore \boxed{x^4-4x^3+x^2+6x+2 \equiv (x-1)^4-5(x-1)^2+6.}$

(ii) Hence $(x-1)^4-5(x-1)^2+6 = 0$

let $(x-1)^2 = u$ [2]

$u^4-5u^2+6 = 0$

$(u-3)(u-2) = 0$

$u = 2, 3$

$\therefore (x-1)^2 = 2$ OR $(x-1)^2 = 3$

$\boxed{x = 1 \pm \sqrt{2}}$ $\boxed{x = 1 \pm \sqrt{3}}$

HSC Maths Ext 1 Assessment Task 1 Question 3

a) (i) $(3, 2) = \left(\frac{2b+a}{3}, \frac{2d+c}{3}\right)$

$\frac{2b+a}{3} = 3$ ($\frac{1}{2}$) $\qquad \frac{2d+c}{3} = 2$ ($\frac{1}{2}$)

$2b+a = 9 \qquad 2d+c = 6$

$\boxed{a = 9-2b}$ ($\frac{1}{2}$) $\qquad \boxed{c = 6-2d}$ ($\frac{1}{2}$)

(ii) $b+3d+15=0$ because $B(b, d)$ lies on the line $x+3y+15=0$ (1)

(iii) $a = 9-2b$ —— (1)

$c = 6-2d$ —— (2)

$4a+3c-30=0$ —— (3)

$b+3d+15=0$ —— (4)

sub (1) & (2) in (3),

$4(9-2b)+3(6-2d)-30=0$

$36-8b+18-6d-30=0$

$8b+6d=24$

$4b+3d=12$ —— (5) (1)

sub (4) in (5),

$4(-3d-15)+3d=12$

$-12d-60+3d=12$

$-9d=72$

$\boxed{d=-8}$ ($\frac{1}{2}$)

$\boxed{b=9}$ ($\frac{1}{2}$)

$\boxed{a=-9}$ ($\frac{1}{2}$)

$\boxed{c=22}$ ($\frac{1}{2}$)

b). (i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At P, $m_T = \frac{2at}{2a} = t$

$m_N = -\frac{1}{t}$ (1)

The eqⁿ of normal at P,

$y - at^2 = -\frac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty = at^3 + 2at$ — (1) (1)

(ii) The eqⁿ of SQ,

$m_{SQ} = t$, $S(0, a)$

$y - a = t(x - 0)$

$y = tx + a$ — (2)

Sub (2) in (1),

$x + t(tx + a) = at^3 + 2at$

$x + t^2x + at = at^3 + 2at$

$(1+t^2)x = at^3 + at$

$(1+t^2)x = at(t^2+1)$ since $t^2+1 \neq 0$

$\boxed{x = at}$ (1)

$y = at(t) + a$

$y = at^2 + a$

$\boxed{y = a(t^2+1)}$ $\therefore Q(at, a(t^2+1))$ (1)

b) (iii) $x = at$, $y = a(t^2+1)$

$y = a\left(\frac{x^2}{a^2} + 1\right)$

$ay = x^2 + a^2$ (1)

$x^2 = a(y-a)$ which is a parabola with focal length $\frac{a}{4}$. (1)

(iv) $N(0, at^2)$
$S(0, a)$
$Q(at, a(t^2+1))$
$P(2at, at^2)$

$ON = at^2$ ($\frac{1}{2}$)

$SP = PM$ (By definition)
$= at^2 + a$
$= a(t^2+1)$ ($\frac{1}{2}$)

$SQ = \sqrt{(at-0)^2 + (at^2+a-a)^2}$

$= \sqrt{a^2t^2 + a^2t^4}$

$= \sqrt{a^2t^2(1+t^2)}$

$= at\sqrt{(t^2+1)}$

$SQ^2 = a^2t^2(t^2+1)$ (1)

$ON \times SP = at^2 \times a(t^2+1)$

$= a^2t^2(t^2+1)$

$= SQ^2$ (1)

Year 11 Maths Ext 1 Nov 2014-11-22

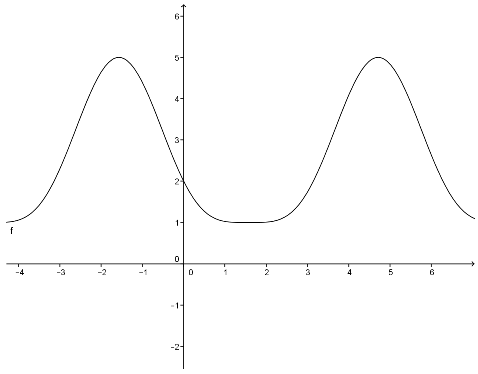
Question 4

(a) (i) $y = \sin^2 x - 2\sin x + 2$

$= (\sin x - 1)^2 + 1$

Considering min and max values of $(\sin x - 1)^2$,

Range: $1 \le y \le 5$ [3]

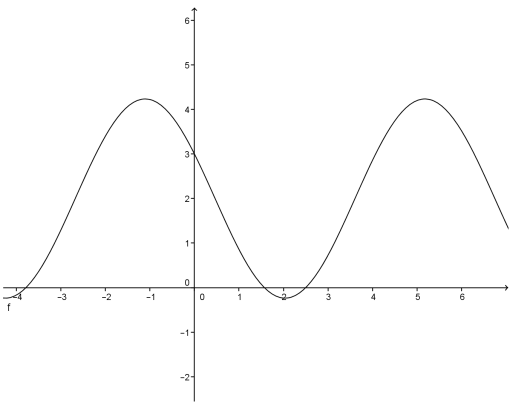


(ii) $y = \cos x - 2\sin x + 2$

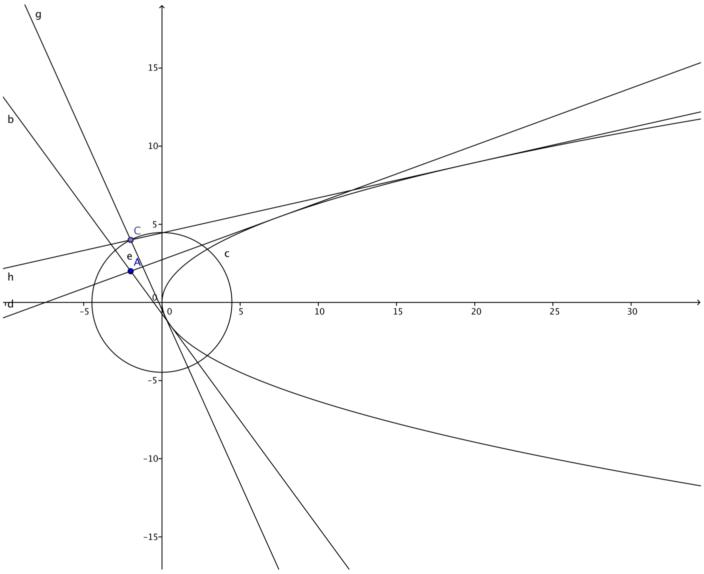
$= \sqrt{5}\left(\cos(x + \alpha)\right) + 2$

Considering max and min values of cosine,

Range: $2 - \sqrt{5} \le y \le 2 + \sqrt{5}$ [2]



(b)



(i) Curve: $y^2 = 4x$ Line: $y = mx + c$

Tangent:
$$(mx+c)^2 = 4x$$
$$m^2x^2 + 2mcx + c^2 - 4x = 0$$
$$m^2x^2 + (2mc-4)x + c^2 = 0$$

For the line to be a tangent, discriminant is zero.
$$(2mc-4)^2 - 4m^2c^2 = 0$$
$$[(2mc-4)-2mc][(2mc-4)+2mc] = 0$$
$$-4[(2mc-4)+2mc] = 0$$
$$4mc - 4 = 0$$
$$mc = 1$$
$$\therefore c = \frac{1}{m}$$ [2]

(ii)
$$y = mx + c$$
$$k = mh + \frac{1}{m}$$
$$mk = m^2h + 1$$
$$\therefore hm^2 - mk + 1 = 0$$ [2]

(iii) Solutions using quadratic formula
$$m = \frac{-k \pm \sqrt{k^2 - 4h}}{2h}$$

Thus $\left(m_1 - m_2\right) = \dfrac{-k + \sqrt{k^2 - 4h}}{2h} - \dfrac{-k - \sqrt{k^2 - 4h}}{2h}$

$$= \frac{2\sqrt{k^2 - 4h}}{2h}$$

$$\left(m_1 - m_2\right)^2 = \frac{k^2 - 4h}{h^2}$$ [2]

(iv) $\left(\dfrac{1}{s_1} - \dfrac{1}{s_2}\right)^2 = \left(\dfrac{s_2 - s_1}{s_1 s_2}\right)^2 = 8$

From $hm^2 - mk + 1 = 0$, $\alpha\beta = \dfrac{c}{a} = \dfrac{1}{h}$

$$\frac{\left(s_1 - s_2\right)^2}{(s_1 s_2)^2} = \frac{\dfrac{k^2 - 4m}{h^2}}{\left(\dfrac{1}{h^2}\right)^2}$$

$$= k^2 - 4h$$

$\therefore k^2 - 4h = 8$

But Q is $\left(h, k\right)$, and $h^2 + k^2 = 20$, so

$h^2 + \left(8 + 4h\right) = 20$, and hence

$h^2 + 4h - 12 = 0$

$\therefore h = -6$ or 2

On substitution in the equation of the circle, -6 is extraneous.
Thus $h = 2, k = \pm 4$.

Hence Q is $\left(2, 4\right)$ or $\left(2, -4\right)$. [2]

(v) If θ is the angle between two lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Here $\tan\theta = \tan 45^o = 1$

Hence $1 = \left(\dfrac{m_1 - m_2}{1 + m_1 m_2}\right)^2$

$$= \frac{k^2 - 4h}{h^2} \div \left(1 + 2m_1m_2 + (m_1m_2)^2\right)$$

$$= \frac{k^2 - 4h}{h^2} \times \frac{1}{1 + 2\left(\frac{1}{h}\right) + \frac{1}{h^2}}$$

$$= \frac{k^2 - 4h}{h^2 + 2h + 1}$$

$\therefore h^2 + 2h + 1 = k^2 - 4h$

ie $(h+3)^2 - k^2 = 8$

On renaming the variables, the locus is

$(x+3)^2 - y^2 = 8$ (an Hyperbola, conjugate axis x=-3) [2]